

# The largest eigenvalue distribution of the Laguerre unitary ensemble

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## Abstract

We study the probability that all eigenvalues of the Laguerre unitary ensemble of  $n$  by  $n$  matrices are in  $(0, t)$ , i.e., the largest eigenvalue distribution. Associated with this probability, in the ladder operator approach for orthogonal polynomials, there are recurrence coefficients, namely  $\alpha_n(t)$  and  $\beta_n(t)$ , as well as three auxiliary quantities, denoted by  $r_n(t)$ ,  $R_n(t)$  and  $\sigma_n(t)$ . We establish the second order differential equations for both  $\beta_n(t)$  and  $r_n(t)$ . By investigating the soft edge scaling limit when  $\alpha = O(n)$  as  $n \rightarrow \infty$  or  $\alpha$  is finite, we derive a  $P_{II}$ , the  $\sigma$ -form, and the asymptotic solution of the probability. In addition, we develop differential equations for orthogonal polynomials  $P_n(z)$  corresponding to the largest eigenvalue distribution of LUE and GUE with  $n$  finite or large. For large  $n$ , asymptotic formulas are given near the singular points of the ODE. Moreover, we are able to deduce a particular case of Chazy's equation for  $\varrho(t) = \Xi'(t)$  with  $\Xi(t)$  satisfying the  $\sigma$ -form of  $P_{IV}$  or  $P_V$ .

## 1 Introduction

A unitary ensemble is well defined for Hermitian matrices  $M = (M_{ij})_{n \times n}$  with probability density

$$p(M)dM \propto e^{-\text{tr } v(M)} \text{vol}(dM), \quad \text{vol}(dM) = \prod_{i=1}^n dM_{ii} \prod_{1 \leq j < k \leq n} d(\text{Re} M_{jk}) d(\text{Im} M_{jk}). \quad (1.1)$$

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Here  $v(M)$  is a matrix function [17] defined via Jordan canonical form and  $\text{vol}(\text{dM})$  is called the volume element [18]. The joint probability density function of the eigenvalues  $\{x_j\}_{j=1}^n$  of this unitary ensemble is given in [22] by

$$\frac{1}{D_n(a, b)} \frac{1}{n!} \prod_{1 \leq j < k \leq n} |x_k - x_j|^2 \prod_{j=1}^n w(x_j), \quad (1.2a)$$

where  $D_n(a, b)$  is the normalization constant which reads

$$D_n(a, b) = \frac{1}{n!} \int_{[a, b]^n} \prod_{1 \leq j < k \leq n} |x_k - x_j|^2 \prod_{j=1}^n w(x_j) dx_j, \quad (1.2b)$$

and  $w(x) = e^{-v(x)}$  is a positive weight function supported on  $[a, b]$  with finite moments

$$\mu_k := \int_a^b x^k w(x) dx, \quad k = 0, 1, 2, \dots$$

It is shown, in [22], that  $D_n(a, b)$  can be evaluated as the determinant of the Hankel (or moment) matrix, that is,

$$D_n(a, b) = \det (\mu_{i+j})_{i,j=0}^{n-1}.$$

A unitary ensemble is called the Laguerre unitary ensemble (LUE) if in (1.1)

$$v(x) = x - \alpha \ln x,$$

or, what amounts to the same thing, in (1.2)

$$w(x) = x^\alpha e^{-x}, \quad x \in [0, \infty), \quad \alpha > 0.$$

A special case of LUE is  $M = XX^*$  and  $\alpha = p - n$ , where  $X = X_1 + iX_2$  is an  $n \times p$  ( $n \leq p$ ) random matrix with each element of  $X_1$  and  $X_2$  chosen independently as a Gaussian random variable, see [13, 14, 15, 19].

Denote by  $\mathbb{P}(n, t)$  the probability that the largest eigenvalue in LUE is not larger than  $t$ , then

$$\mathbb{P}(n, t) = \frac{D_n(t)}{D_n(0, \infty)},$$

where  $D_n(t) := D_n(0, t)$ . Tracy and Widom [27] have obtained the Jimbo-Miwa-Okamoto (J-M-O)  $\sigma$ -form [20, 23] of  $P_V$  for

$$\sigma_n(t) := t \frac{d}{dt} \ln \mathbb{P}(n, t)$$

by making use of the Fredholm determinant. Basor and Chen [1] have derived the same  $\sigma$ -form by studying the Hankel determinant  $D_n(t)$  with the help of the ladder operators related to orthogonal polynomials. In their work, another four quantities associated with  $\mathbb{P}(n, t)$  are considered, i.e.  $\alpha_n(t)$ ,  $\beta_n(t)$ ,  $r_n(t)$  and  $R_n(t)$ , and the relationships between them are established. In addition, a  $P_V$  is derived for  $R_n(t)$  (or  $\alpha_n(t)$ ). Based on these results, we obtain in this paper the second order differential equation for  $\beta_n(t)$  as well as  $r_n(t)$ .

The soft edge scaling limit of the smallest eigenvalue distribution on  $(t, \infty)$  in LUE with  $\alpha = \mu n = O(n)$  and  $t = (\sqrt{\mu+1} - 1)^2 n - \frac{(\sqrt{\mu+1}-1)^{4/3}}{(\mu+1)^{1/6}} n^{1/3} s$  is analyzed in [24]. Concerning the largest eigenvalue distribution, we show that for  $\alpha = O(n)$  or finite, and

$$t = c_1 n + c_2 n^{1/3} s, \quad \sigma(s) := \frac{c_2}{c_1} \lim_{n \rightarrow \infty} n^{-2/3} \sigma_n(t)$$

where

$$c_1 = \left( \sqrt{\mu+1} + 1 \right)^2, \quad c_2 = \frac{(\sqrt{\mu+1} + 1)^{4/3}}{(\mu+1)^{1/6}}, \quad \mu = \begin{cases} \frac{\alpha}{n}, & \alpha = O(n) \\ 0, & \alpha \text{ is finite} \end{cases},$$

the aforementioned  $\sigma$ -form of  $P_V$  reduces down to the same  $\sigma$ -form of  $P_{II}$  as presented in [24]. The  $P_V$ , the ODEs for  $\beta_n(t)$  and  $r_n(t)$  can likewise be reduced to a  $P_{II}$ . According to the ODE for  $\sigma(s)$ , we are able to provide the behavior of  $\mathbb{P}(n, t)$  for large  $n$  when  $s \rightarrow \infty$  or  $s \rightarrow -\infty$ .

By means of the ladder operators valid for the orthogonal polynomials  $P_n(z)$  associated with the general weight function  $w(x) = e^{-v(x)}$ , we deal with our problem and the largest eigenvalue distribution of GUE, and show that the corresponding  $\phi_n(z) := e^{-v(z)/2} P_n(z)$  satisfy different second order ODEs for finite  $n$  but the same one for large  $n$ . In the case of large  $n$ , we develop the asymptotic behavior and Taylor expansion of  $\phi_n(z)$  near the singular points of the corresponding ODE. Moreover, a Chazy's equation [11] is derived for  $\varrho(t) := \Xi'(t)$  with  $\Xi(t)$  satisfying the  $\sigma$ -form of  $P_{IV}$  or  $P_V$ , and this result is applied to different ensembles including the largest eigenvalue distribution of LUE and GUE.

This paper is built up as follows. In Section 2, we introduce the ladder operator technique and restate the results of [1] which are used throughout this paper for further derivation. We produce the ODE for  $\beta_n(t)$  and establish a mapping for  $r_n(t)$  and  $R_n(t)$ . The soft edge scaling limit is studied in Section 3. The limiting behavior of  $\phi_n(z)$  in the neighbourhood of the singular points is then presented in Section 4. Finally, Section 5 is devoted to a derivation of Chazy's equations.

## 2 Preliminaries

Monic polynomials  $\{P_n(x)\}$  orthogonal with respect to a generic weight  $w(x)$  on  $[a, b]$  is defined by the relations

$$\int_a^b P_m(x)P_n(x)w(x)dx = h_n\delta_{mn}, \quad m \geq 0, \quad n \geq 0, \quad (2.1)$$

where  $h_n$  is the square of the  $L^2$  norm of the polynomial  $P_n(x)$  and

$$P_n(x) = x^n + p_1(n)x^{n-1} + \cdots + P_n(0). \quad (2.2)$$

An immediate consequence of the orthogonality relation is the three-term recurrence relation [25]

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x), \quad n \geq 0 \quad (2.3)$$

with initial conditions

$$P_0(x) := 1, \quad \beta_0 P_{-1}(x) := 0.$$

Substituting (2.2) into this relation gives rise to

$$\alpha_n = p_1(n) - p_1(n+1), \quad n \geq 0$$

with  $p_1(0) := 0$ , which immediately yields

$$\sum_{j=0}^{n-1} \alpha_j = -p_1(n).$$

From the recurrence relation (2.3) and the orthogonality relation (2.1), we get

$$\beta_n = \frac{h_n}{h_{n-1}}.$$

The lowering and raising ladder operators (see e.g. [6], [7] for a precise statement) are

$$\begin{aligned} \left( \frac{d}{dz} + B_n(z) \right) P_n(z) &= \beta_n A_n(z) P_{n-1}(z), \\ \left( \frac{d}{dz} - B_n(z) - v'(z) \right) P_{n-1}(z) &= -A_{n-1}(z) P_n(z), \end{aligned} \quad (2.4a)$$

with

$$\begin{aligned} A_n(z) &= \frac{P_n^2(y)w(y)}{h_n(y-z)} \Big|_{y=a}^{y=b} + \frac{1}{h_n} \int_a^b \frac{v'(z) - v'(y)}{z-y} P_n^2(y)w(y)dy, \\ B_n(z) &= \frac{P_n(y)P_{n-1}(y)w(y)}{h_{n-1}(y-z)} \Big|_{y=a}^{y=b} + \frac{1}{h_{n-1}} \int_a^b \frac{v'(z) - v'(y)}{z-y} P_n(y)P_{n-1}(y)w(y)dy, \end{aligned} \quad (2.4b)$$

and  $v(z) := -\ln w(z)$ . The compatibility conditions  $(S_1)$ ,  $(S_2)$  and  $(S'_2)$  for the ladder operators, see [8, 14, 21], are given by

$$B_{n+1}(z) + B_n(z) = (z - \alpha_n)A_n(z) - v'(z), \quad (S_1)$$

$$1 + (z - \alpha_n)(B_{n+1}(z) - B_n(z)) = \beta_{n+1}A_{n+1}(z) - \beta_n A_{n-1}(z), \quad (S_2)$$

$$B_n^2(z) + v'(z)B_n(z) + \sum_{j=0}^{n-1} A_j(z) = \beta_n A_n(z)A_{n-1}(z). \quad (S'_2)$$

The discontinuous Laguerre weight

$$w(x) = (A + B\theta(x - t))x^\alpha e^{-x}, \quad A \geq 0, \quad A + B \geq 0, \quad (2.5a)$$

where

$$A + B\theta(x - t) = \begin{cases} A + B, & \text{if } x > t \\ A, & \text{if } x \leq t \end{cases} \quad (2.5b)$$

is investigated in [1], and the case where  $A = 0$  and  $B = 1$  leads to the smallest eigenvalue distribution of LUE on  $(t, \infty)$ . For our problem at hand, which is the case where  $A = 1$  and  $B = -1$ , it is shown that

$$\begin{aligned} A_n(z) &= \frac{R_n(t)}{z - t} + \frac{1 - R_n(t)}{z}, \\ B_n(z) &= \frac{r_n(t)}{z - t} - \frac{r_n(t) + n}{z}, \end{aligned} \quad (2.6)$$

with

$$R_n(t) := -\frac{P_n^2(t, t)}{h_n(t)}t^\alpha e^{-t}, \quad r_n(t) := -\frac{P_n(t, t)P_{n-1}(t, t)}{h_{n-1}(t)}t^\alpha e^{-t},$$

where  $P_j(t, t) := P_j(z, t) |_{z=t}$ . It should be noted that the  $t$  dependence through the weight induces  $t$  dependence of  $P_j(x)$ ,  $h_j$  and their allied quantities. For the sake of brevity, we shall not display the independence on  $t$  for latter discussion unless we have to.

By using the compatibility conditions and taking the derivative of the orthogonality relation (2.1) with respect to  $t$ , in addition to a  $P_V$  and the  $\sigma$ -form, Basor and Chen show that  $r_n$  and  $R_n$  which are closely related to  $\alpha_n$  and  $\beta_n$  satisfy a couple of difference equations. We recall a number of results from [1] and make some remarks for our new observations.

**Proposition 2.1.** *The relations between  $\sigma_n(t) := t \frac{d}{dt} \ln \mathbb{P}(n, t)$  and other quantities are*

$$\begin{aligned}
\sigma_n &= t \frac{d}{dt} \ln D_n = -t \sum_{j=0}^{n-1} R_j \\
&= n(n + \alpha) + p_1(n) \\
&= n(n + \alpha) + tr_n - \beta_n, \\
\sigma'_n &= r_n, \quad t\sigma''_n = tr'_n = \beta'_n.
\end{aligned} \tag{2.7}$$

**Proposition 2.2.** (a) *The quantities  $r_n$  and  $R_n$  satisfy the following coupled Riccati equations:*

$$tr'_n = \left( \frac{1}{R_n} + \frac{1}{R_n - 1} \right) r_n^2 + (2n + \alpha) \frac{R_n}{R_n - 1} r_n + n(n + \alpha) \frac{R_n}{R_n - 1}, \tag{2.8}$$

$$tR'_n = tR_n^2 + (2n + \alpha - t)R_n + 2r_n. \tag{2.9}$$

(b) *The difference equations for  $r_n(t)$  and  $R_n(t)$  read*

$$\begin{aligned}
r_{n+1} + r_n &= (t - 2n - 1 - \alpha - tR_n)R_n, \\
r_n^2 \left( \frac{1}{R_n R_{n-1}} - \frac{1}{R_n} - \frac{1}{R_{n-1}} \right) &= (2n + \alpha)r_n + n(n + \alpha).
\end{aligned}$$

**Remark 1.** The equations in (b) can be rewritten as

$$\begin{aligned}
x_{n+1}x_n &= \frac{y_n^2 - (2n + \alpha)y_n + n(n + \alpha)}{y_n^2}, \\
y_n + y_{n-1} &= -\frac{(-t + 2n - 1 + \alpha)x_n - (2n - 1 + \alpha)}{x_n^2 - 2x_n + 1},
\end{aligned}$$

where

$$x_n := 1 - \frac{1}{R_{n-1}}, \quad y_n := -r_n.$$

This mapping is very similar to (25) in [16] which leads to discrete Painlevé equations.

**Proposition 2.3.** (a) *The recurrence coefficients  $\alpha_n$  and  $\beta_n$  are expressed in terms of  $r_n$  and  $R_n$  as follows:*

$$\begin{aligned}
\alpha_n &= 2n + 1 + \alpha + tR_n, \\
\beta_n &= \frac{1}{1 - R_n} \left( (2n + \alpha)r_n + n(n + \alpha) + \frac{r_n^2}{R_n} \right).
\end{aligned}$$

(b) *The following equation holds*

$$((2n + \alpha)^2 - 4\beta_n) r_n^2 + 2(2n + \alpha)(n(n + \alpha) - \beta_n)r_n + (n(n + \alpha) - \beta_n)^2 - (\beta'_n)^2 = 0. \quad (2.10)$$

**Remark 2.** Equation (2.10) is given at the end of the proof of Theorem 6 in [1]. Solving for  $r_n$  from it, differentiating both sides of the resulting equation with respect to  $t$  and noting that  $\beta'_n = tr'_n$ , we establish the differential equation for  $\beta_n = \beta_n(t)$  :

$$\begin{aligned} & t^2 \left( 2n^2(\alpha + n)^2(\alpha + 2n)^2 - 8n(\alpha + n) (\alpha^2 + 3n^2 + 3\alpha n) \beta_n + 6(\alpha + 2n)^2 \beta_n^2 \right. \\ & \quad \left. - 8\beta_n^3 + 2 \left( (\alpha + 2n)^2 - 4\beta_n \right) \beta_n'^2 + \left( (\alpha + 2n)^2 - 4\beta_n \right)^2 \beta_n'' \right)^2 \\ & = \left( (\alpha + 2n)^4 - \alpha^2(\alpha + 2n)t - 8(\alpha + 2n)^2 \beta_n + 16\beta_n^2 \right)^2 \\ & \quad \cdot \left( 4\beta_n (n(\alpha + n) - \beta_n)^2 + \left( (\alpha + 2n)^2 - 4\beta_n \right) \beta_n'^2 \right). \end{aligned} \quad (2.11)$$

**Proposition 2.4.** (a) *The quantity*

$$S_n(t) := 1 - \frac{1}{R_n(t)}$$

*satisfies the following equation*

$$\begin{aligned} S_n'' &= \left( \frac{1}{2S_n} + \frac{1}{S_n - 1} \right) (S_n')^2 - \frac{1}{t} S_n' - \frac{(S_n - 1)^2}{t^2} \left( \frac{\alpha^2}{2} \cdot \frac{1}{S_n} \right) \\ &\quad + (2n + 1 + \alpha) \frac{S_n}{t} - \frac{1}{2} \frac{S_n(S_n + 1)}{S_n - 1}, \end{aligned} \quad (2.12)$$

*which is a  $P_V$  [22] with*

$$\alpha = 0, \quad \beta = -\frac{\alpha^2}{2}, \quad \gamma = 2n + 1 + \alpha, \quad \delta = -\frac{1}{2}.$$

(b) *The differential equation for  $\sigma_n$  reads*

$$(t\sigma_n'')^2 = (\sigma_n - (t - 2n - \alpha)\sigma_n')^2 + 4\sigma_n'^2(\sigma_n - t\sigma_n' - n(n + \alpha)), \quad (2.13a)$$

*which is the Jimbo-Miwa-Okamoto  $\sigma$ -form [20, 23] of  $P_V$  (see (5.1) below) with*

$$\nu_1 = 0, \quad \nu_2 = n, \quad \nu_3 = n + \alpha. \quad (2.13b)$$

**Remark 3.** Combining (2.7), (2.9) and (2.10) gives

$$\sigma_n = \frac{\alpha^2}{4} \cdot \frac{R_n}{1 - R_n} - \frac{1}{4}(4n + 2\alpha - t)tR_n - \frac{1}{4}t^2R_n^2 - \frac{1}{4} \frac{t^2(R_n')^2}{R_n(1 - R_n)} \quad (2.14)$$

$$= -\frac{\alpha^2}{4} \cdot \frac{1}{S_n} + \frac{t}{4} \cdot \frac{4n + 2\alpha - t}{S_n - 1} - \frac{1}{4} \cdot \frac{t^2}{(S_n - 1)^2} + \frac{1}{4} \cdot \frac{t^2(S_n')^2}{(S_n - 1)^2 S_n}. \quad (2.15)$$

This is the desired relation for  $\sigma_n$  and  $S_n$  as it was demonstrated in [20].

### 3 Soft edge scaling limit: $P_{II}$ , the $\sigma$ -form of $P_{II}$ and the tail behavior of $\mathbb{P}(n, t)$

For large  $n$ , we define

$$t = c_1 n + c_2 n^{1/3} s,$$

where

$$c_1 = \left( \sqrt{\mu + 1} + 1 \right)^2, \quad c_2 = \frac{(\sqrt{\mu + 1} + 1)^{4/3}}{(\mu + 1)^{1/6}}, \quad \mu = \begin{cases} \frac{\alpha}{n}, & \alpha = O(n) \\ 0, & \alpha \text{ is finite} \end{cases}.$$

In the case of finite  $\alpha$ , we have

$$c_1 = 4, \quad c_2 = 4^{2/3}, \quad t = 4n + 4^{2/3} n^{1/3} s.$$

We also use the following symbols:

$$\begin{aligned} y(s) &:= \frac{c_2}{c_1} \lim_{n \rightarrow \infty} \frac{S_n(t)}{n^{2/3}}, \\ \sigma(s) &:= \frac{c_2}{c_1} \lim_{n \rightarrow \infty} \frac{\sigma_n(t)}{n^{2/3}}, \\ r(s) &:= \frac{c_2^2}{c_1} \lim_{n \rightarrow \infty} \frac{r_n(t)}{n^{1/3}}. \end{aligned}$$

Combining (2.7) with  $\sigma'_n(t) = r_n(t)$ , and (2.9) with (2.10), we obtain

$$r(s) = \sigma'(s) = \frac{c_2^2}{c_1^2} \lim_{n \rightarrow \infty} \frac{\beta_n(t) - n(n + \alpha)}{n^{4/3}} \quad (3.1)$$

$$= \frac{c_1}{c_2} \lim_{n \rightarrow \infty} n^{2/3} R_n(t) = \frac{1}{c_2} \lim_{n \rightarrow \infty} \frac{\alpha_n(t) - 2n - \alpha}{n^{1/3}}. \quad (3.2)$$

Based on the results for finite  $n$  presented in the previous sections, we are able to find the relations and ODEs for the above four quantities. The equation for  $\sigma(s)$  indicates the behavior of  $\mathbb{P}(n, t)$  when both  $n$  and  $|s|$  large.

A  $P_{II}$  and the  $\sigma$ -form of  $P_{II}$  are established in [24] for the smallest eigenvalue distribution of LUE with  $\alpha = O(n)$ . The analysis there is done directly on the associated  $P_V$  equation. For our problem, we shall now present a further and more detailed investigation of  $\sigma(s)$ .



**Theorem 3.1.** (i) *The variables  $\sigma(s)$ ,  $y(s)$ , and  $r(s)$  are connected by the relations*

$$\sigma(s) = -\frac{s}{y(s)} - \frac{1}{y^2(s)} + \frac{(y'(s))^2}{4y^3(s)} \quad (3.3)$$

$$= -\frac{r'(s)^2}{4r(s)} - r(s)^2 + sr(s), \quad (3.4)$$

$$r(s) = -\frac{1}{y(s)}. \quad (3.5)$$

(ii) *The following equation is valid*

$$y''(s) = \frac{3}{2} \cdot \frac{y'(s)^2}{y(s)} - 2sy(s) - 4, \quad (3.6)$$

*in which we introduce*

$$y(s) = w^{-2}(s),$$

*so that  $w(s)$  satisfies the  $P_{II}$  with  $\alpha = 0$ , namely*

$$w''(s) = 2w^3(s) + sw(s).$$

(iii) *The equation for  $\sigma(s)$  reads*

$$\sigma''(s)^2 + 4\sigma'(s) (\sigma'(s)^2 - s\sigma'(s) + \sigma(s)) = 0, \quad (3.7)$$

*which can be brought into the  $\sigma$ -form of  $P_{II}$  with  $\theta = 0$  by making the replacements  $s \rightarrow -2^{-1/3}s$  and  $\sigma(s) \rightarrow -2^{1/3}\sigma(s)$ . Moreover, for  $r(s)$ , we have*

$$r''(s) = \frac{1}{2} \cdot \frac{r'(s)^2}{r(s)} + 2sr(s) - 4r^2(s). \quad (3.8)$$

*Proof.* (3.3) can be verified by (2.15). To be specific, substituting  $\sigma_n(t)$  by  $\frac{c_1}{c_2}n^{2/3}\sigma(s)$ , and  $S_n(t)$  by  $\frac{c_1}{c_2}n^{2/3}y(s)$  in (2.15), we obtain (3.3) as the only retained leading order term when  $n \rightarrow \infty$ . (3.5) is a direct consequence of the first expression in (3.2), which combined with (3.3) results in (3.4). Finally, equation (3.6) arises from (2.12), (3.7) from (2.13a), and (3.8) from (2.11) (as well as from (5.4), see below). In case (3.8) we take into account (3.1).  $\square$

**Remark 4.** We can prove (3.6)-(3.8) in another way, by use of (3.3)-(3.5) and  $\sigma'(s) = r(s)$ . Indeed, differentiating both sides of (3.3), in view of (3.5), we get (3.6). Replacing  $r(s)$  by  $\sigma'(s)$  and  $r'(s)$  by  $\sigma''(s)$  in (3.4) yields (3.7). Solve for  $\sigma(s)$  from (3.7), then differentiation gives (3.8).

Now we go ahead with the evaluation of  $\mathbb{P}(n, t)$ . The definitions of  $\sigma_n(t)$  and  $\sigma(s)$  imply

$$\sigma(s) = \frac{d}{ds} \ln \hat{\mathbb{P}}(s),$$

where  $\hat{\mathbb{P}}(s) := \lim_{n \rightarrow \infty} \mathbb{P}(n, c_1 n + c_2 n^{1/3} s)$ . To obtain the expansion of  $\sigma(s)$  as  $s \rightarrow -\infty$ , we assume

$$\sigma(s) = \lambda_2 s^2 + \lambda_1 s + \sum_{k=0}^{\infty} \frac{d_k}{s^k}.$$

Substituting this into (3.7) yields

$$\sigma(s) = \frac{s^2}{4} - \frac{1}{8s} + \frac{9}{64s^4} - \frac{189}{128s^7} + \frac{21663}{512s^{10}} + O\left(\frac{1}{s^{13}}\right),$$

from which follows

$$\begin{aligned} \hat{\mathbb{P}}(s) &= \iota_1 \frac{\exp\left(\frac{s^3}{12}\right)}{(-s)^{1/8}} \exp\left(-\frac{3}{64s^3} + \frac{63}{256s^6} - \frac{2407}{512s^9} + O\left(\frac{1}{s^{12}}\right)\right) \\ &= \iota_1 \frac{\exp\left(\frac{s^3}{12}\right)}{(-s)^{1/8}} \left(1 - \frac{3}{2^6 s^3} + \frac{2025}{2^{13} s^6} - \frac{2470825}{2^{19} s^9} + \frac{26389914075}{2^{27} s^{12}} + O\left(\frac{1}{s^{12}}\right)\right), \end{aligned}$$

where the second equality results from the Taylor series for the exponential function. Here  $\iota_1$  is the normalization constant and is given in [28] by

$$\iota_1 = 2^{1/24} e^{\zeta'(-1)},$$

where  $\zeta'(-1)$  is the derivative of the Riemann zeta function evaluated at -1. The above result agrees with (1.19) in [26], since  $w(z)$  plays the same role as  $q(s; \lambda)$  there. In fact, because of (3.5) and  $y(s) = w^{-2}(s)$ , we have

$$\sigma'(s) = -w^2(s).$$

As  $s \rightarrow \infty$ , to continue, we write

$$\hat{\mathbb{P}}(s) = 1 - \varepsilon f(s),$$

where  $\varepsilon > 0$  is sufficiently small and  $f(s) > 0$ , then

$$\sigma(s) = \frac{\varepsilon f'(s)}{1 - \varepsilon f(s)}.$$

Hence from the left hand side of (3.7) there follows a quadratic polynomial in  $\varepsilon$  and the constant term of which has to be 0, namely

$$f^{(3)}(s)^2 - 4s f''(s)^2 + 4f'(s) f''(s) = 0.$$

If we define

$$h(s) := \frac{f'(s)}{f(s)} = \frac{d}{ds} \ln f(s),$$

then

$$(h''(s) + 3h(s)h'(s) + h^3(s))^2 - 4s(h'(s) + h^2(s))^2 + 4h(s)(h'(s) + h^2(s)) = 0.$$

Suppose

$$h(s) = \lambda_0 s^{1/2} + \sum_{k=0}^{\infty} \frac{\nu_k}{s^{k/2}},$$

then the coefficients can be determined by use of the preceding equation for  $h(s)$  and we obtain

$$h(s) = -2s^{1/2} - \frac{3}{2s} + \frac{35}{16s^{5/2}} + O\left(\frac{1}{s^4}\right),$$

which implies

$$\hat{\mathbb{P}}(s) = 1 - \iota_2 \frac{\exp\left(-\frac{4}{3}s^{3/2}\right)}{s^{3/2}} \left(1 - \frac{35}{24s^{3/2}} + O\left(\frac{1}{s^3}\right)\right).$$

Referring to [28], we see that  $\iota_2 = \frac{1}{16\pi}$ .

## 4 The behavior of $z^{\alpha/2}e^{-z/2}P_n(z)$ on $(0, t)$ and of $e^{-z^2/2}P_n(z)$ on $(-\infty, t)$ for finite $n$ and large $n$

We now turn our attention to the ladder operators given by (2.4) to develop the differential equation for orthogonal polynomials  $P_n(z)$  defined by (2.1) and (2.2). Eliminating  $P_{n-1}(z)$  from the lowering and raising operators, incorporated with  $(S'_2)$ , produces

$$P_n'' - \left(\frac{A'_n}{A_n} + v'\right)P_n' + \left(B'_n - \frac{A'_n}{A_n}B_n + \sum_{j=0}^{n-1} A_j\right)P_n = 0, \quad (4.1)$$

which is presented in [1] and [3]. To continue, we set, with suitable continuation in  $z$ ,

$$P_n(z) = e^{v(z)/2}\phi_n(z), \quad v(z) = -\ln w(z),$$

and, in light of (4.1), establish

$$\phi_n'' - \frac{A'_n}{A_n}\phi_n' + \left(B'_n - \frac{A'_n}{A_n}\left(B_n + \frac{v'}{2}\right) + \sum_{j=0}^{n-1} A_j + \frac{v''}{2} - \frac{v'^2}{4}\right)\phi_n = 0. \quad (4.2)$$

Below we will apply the above equation to the largest eigenvalue distribution of LUE on  $(0, t)$  and of GUE on  $(-\infty, t)$ . For the Laguerre case, that is,

$$\phi_n(z) = z^{\alpha/2} e^{-z/2} P_n(z),$$

using the same notations as in Section 2 for finite  $n$  and as in Section 3 for large  $n$ , we have the following results :

**Theorem 4.1.** (i) *For finite  $n$*

$$\begin{aligned} \phi_n''(z) + \left( \frac{1}{z} + \frac{1}{z-t} - \frac{1}{z-t+tR_n(t)} \right) \phi_n'(z) \\ + \left( \frac{-\frac{1}{4}\alpha^2}{z^2} + \frac{\frac{1}{2}(2n+\alpha+1) - \kappa_1(t) - \kappa_2(t)}{z} + \frac{\kappa_1(t)}{z-t} + \frac{\kappa_2(t)}{z-t+tR_n(t)} - \frac{1}{4} \right) \phi_n(z) = 0, \end{aligned} \quad (4.3a)$$

where

$$\kappa_1(t) = \frac{1}{2}R_n(t) - \frac{R_n'(t)}{2R_n(t)} - \frac{\sigma_n(t)}{t}, \quad \kappa_2(t) = \frac{R_n'(t)}{2R_n(t)} + \frac{R_n'(t)}{2(1-R_n(t))}, \quad (4.3b)$$

and  $\sigma_n(t)$  can be expressed in terms of  $R_n(t)$  by using (2.14).

(ii) *For  $z = c_1 n + c_2 n^{1/3} z^*$ , and  $\phi(z^*) := \lim_{n \rightarrow \infty} \phi_n(z)$*

$$\begin{aligned} \phi''(z^*) + \left( \frac{1}{z^* - s} - \frac{1}{z^* - s + r(s)} \right) \phi'(z^*) \\ + \left( \frac{\frac{r'(s)^2}{4r(s)} - \frac{r'(s)}{2r(s)} - sr(s) + r^2(s)}{z^* - s} + \frac{\frac{r'(s)}{2r(s)}}{z^* - s + r(s)} - z^* \right) \phi(z^*) = 0. \end{aligned} \quad (4.4)$$

*Proof.* (i) Substituting (2.6) and  $v(z) = z - \alpha \ln z$  in (4.2) furnishes

$$\begin{aligned} \phi_n''(z) + \left( \frac{1}{z} + \frac{1}{z-t} - \frac{1}{z-t+tR_n(t)} \right) \phi_n'(z) \\ + \left( \frac{-\frac{1}{4}\alpha^2}{z^2} + \frac{\frac{1}{2}(2n+\alpha+1) - \kappa_1(t) - \kappa_2(t)}{z} + \frac{\kappa_1(t)}{z-t} + \frac{\kappa_2(t)}{z-t+tR_n(t)} - \frac{1}{4} \right) \phi_n(z) = 0, \end{aligned}$$

where

$$\begin{aligned} \kappa_1(t) &= -\frac{1}{t} \left( \frac{r_n(t)}{R_n(t)} + \frac{\alpha}{2} + n \right) + \sum_{j=0}^{n-1} R_j(t) + \frac{1}{2}, \\ \kappa_2(t) &= \frac{1}{t-tR_n(t)} \left( \frac{r_n(t)}{R_n(t)} + \frac{\alpha}{2} + n \right) - \frac{1}{2}. \end{aligned}$$

Applying (2.9) to cancel  $r_n(t)$  in  $\kappa_1(t)$  and  $\kappa_2(t)$ , and taking into consideration  $\sum_{j=0}^{n-1} R_j(t) = -\frac{\sigma_n(t)}{t}$ , we get (4.3b) and hence (4.3).

(ii) Using the first equality of (3.2) and the definition of  $\sigma(s)$ , we can show that

$$c_2 \lim_{n \rightarrow \infty} n^{1/3} \kappa_1(t) = -\frac{r'(s)}{2r(s)} - \sigma(s), \quad c_2 \lim_{n \rightarrow \infty} n^{1/3} \kappa_2(t) = \frac{r'(s)}{2r(s)}.$$

Denote the coefficient of  $\phi_n(z)$  in (4.3a) by  $C_L$ , that is,

$$C_L = \frac{-\frac{1}{4}\alpha^2}{z^2} + \frac{\frac{1}{2}(2n + \alpha + 1)}{z} - \frac{1}{4} - \frac{\kappa_1(t) + \kappa_2(t)}{z} + \frac{\kappa_1(t)}{z - t} + \frac{\kappa_2(t)}{z - t + tR_n(t)}.$$

Then, for  $z = c_1n + c_2n^{1/3}z^*$  and  $t = c_1n + c_2n^{1/3}s$ , we have

$$\begin{aligned} c_2^2 \lim_{n \rightarrow \infty} n^{2/3} C_L &= c_2^2 \lim_{n \rightarrow \infty} n^{2/3} \left( \frac{-\frac{1}{4}\alpha^2}{z^2} + \frac{\frac{1}{2}(2n + \alpha + 1)}{z} - \frac{1}{4} \right) \\ &\quad - c_2^2 \lim_{n \rightarrow \infty} n^{2/3} \frac{\kappa_1(t) + \kappa_2(t)}{z} + c_2^2 \lim_{n \rightarrow \infty} n^{2/3} \frac{\kappa_1(t)}{z - t} \\ &\quad + c_2^2 \lim_{n \rightarrow \infty} n^{2/3} \frac{\kappa_2(t)}{z - t + tR_n(t)} \\ &= -z^* - 0 + \frac{-\frac{r'(s)}{2r(s)} - \sigma(s)}{z^* - s} + \frac{\frac{r'(s)}{2r(s)}}{z^* - s + r(s)} \\ &= \frac{-\frac{r'(s)}{2r(s)} - \sigma(s)}{z^* - s} + \frac{\frac{r'(s)}{2r(s)}}{z^* - s + r(s)} - z^*. \end{aligned}$$

Noting that, for  $\phi(z^*) := \lim_{n \rightarrow \infty} \phi_n(z)$ ,

$$\phi'(z^*) = c_2 \lim_{n \rightarrow \infty} n^{1/3} \phi'_n(z), \quad \phi''(z^*) = c_2^2 \lim_{n \rightarrow \infty} n^{2/3} \phi''_n(z),$$

according to (4.3a) multiplied by  $c_2^2 n^{2/3}$ , as  $n \rightarrow \infty$ , we conclude that (4.4) is valid. □

Now we consider the largest eigenvalue distribution of GUE on  $(-\infty, t)$ . Let  $P_n(x)$  be monic polynomials orthogonal with respect to the deformed Hermite weight with one jump

$$w(x) = 2e^{-x^2} (1 - \theta(x - t)) = \begin{cases} 0, & \text{if } x > t \\ 2e^{-x^2}, & \text{if } x \leq t \end{cases},$$

namely

$$\int_{-\infty}^{\infty} P_m(x) P_n(x) w(x) dx = 2h_n(t) \delta_{mn},$$

or, what amounts to the same thing,

$$\int_{-\infty}^t P_m(x) P_n(x) e^{-x^2} dx = h_n(t) \delta_{mn}.$$

Consequently, the results in [10] with  $\beta = -2$  there in the weight function are valid for our  $P_n(x)$  which corresponds to the largest eigenvalue distribution of GUE on  $(-\infty, t)$ . To begin with,

$$A_n(z) = 2 \left( \frac{\alpha_n(t)}{z-t} + 1 \right), \quad B_n(z) = \frac{r_n(t)}{z-t}, \quad (4.5)$$

where  $\alpha_n(t)$  is the recurrence coefficient, that is,

$$zP_n(z) = P_{n+1}(z) + \alpha_n(t)P_n(z) + \beta_n(t)P_{n-1}(z),$$

and  $r_n(t)$  is defined by

$$r_n(t) := -\frac{P_n(t, t)P_{n-1}(t, t)}{h_{n-1}(t)}e^{-t^2}.$$

Based on the results in [10], for

$$\phi_n(z) = e^{-z^2/2}P_n(z),$$

we obtain the following analogue of the preceding theorem:

**Theorem 4.2.** (i) *For finite  $n$*

$$\begin{aligned} \phi_n''(z) + \left( \frac{1}{z-t} - \frac{1}{z-t+\alpha_n(t)} \right) \phi_n'(z) \\ + \left( \frac{\kappa_3(t)}{z-t} + \frac{\kappa_4(t)}{z-t+\alpha_n(t)} + 2n+1-z^2 \right) \phi_n(z) = 0, \end{aligned} \quad (4.6a)$$

where

$$\kappa_3(t) = \frac{(\alpha_n'(t))^2}{4\alpha_n(t)} - \frac{\alpha_n'(t)}{2\alpha_n(t)} - \alpha_n^3(t) + 2t\alpha_n^2(t) + (-t^2 + 2n+1)\alpha_n(t), \quad \kappa_4(t) = \frac{\alpha_n'(t)}{2\alpha_n(t)}, \quad (4.6b)$$

and  $\alpha_n(t)$  satisfies

$$\alpha_n''(t) = \frac{(\alpha_n'(t))^2}{2\alpha_n(t)} + 6\alpha_n^3(t) - 8t\alpha_n^2(t) + 2(t^2 - 2n - 1)\alpha_n(t). \quad (4.7)$$

(ii) *For  $z = \sqrt{2n} + \frac{z^*}{\sqrt{2n^{1/6}}}$ ,  $t = \sqrt{2n} + \frac{s}{\sqrt{2n^{1/6}}}$ , and  $\phi(z^*) := \lim_{n \rightarrow \infty} \phi_n(z)$*

$$\begin{aligned} \phi''(z^*) + \left( \frac{1}{z^* - s} - \frac{1}{z^* - s + u(s)} \right) \phi'(z^*) \\ + \left( \frac{\frac{(u'(s))^2}{4u(s)} - \frac{u'(s)}{2u(s)} - su(s) + u^2(s)}{z^* - s} + \frac{\frac{u'(s)}{2u(s)}}{z^* - s + u(s)} - z^* \right) \phi(z^*) = 0, \end{aligned} \quad (4.8)$$

where  $u(s) := \lim_{n \rightarrow \infty} \sqrt{2n^{1/6}}\alpha_n(t)$  is a solution of (3.8).

*Proof.* (i) Substitution of (4.5) in (4.2) leads to

$$\begin{aligned} \phi_n''(z) + \left( \frac{1}{z-t} - \frac{1}{z-t+\alpha_n(t)} \right) \phi_n'(z) \\ + \left( \frac{-\frac{r_n(t)}{\alpha_n(t)} + t + 2 \sum_{j=0}^{n-1} \alpha_j(t)}{z-t} + \frac{\frac{r_n(t)}{\alpha_n(t)} - t + \alpha_n(t)}{z-t+\alpha_n(t)} + 2n+1-z^2 \right) \phi_n(z) = 0. \end{aligned} \quad (4.9)$$

The following results are established in [10] (see (22), (24), (26), (28) and (31)),

$$r_n^2(t) = 2(n+r_n(t))\alpha_n(t)\alpha_{n-1}(t), \quad (4.10a)$$

$$r_n'(t) = 2(n+r_n(t))(\alpha_{n-1}(t) - \alpha_n(t)), \quad (4.10b)$$

$$r_n(t) = \alpha_n(t)(t - \alpha_n(t)) + \frac{1}{2}\alpha_n'(t), \quad (4.10c)$$

$$-2 \sum_{j=0}^{n-1} \alpha_j(t) = 2tr_n(t) - 2(n+r_n(t))(\alpha_n(t) + \alpha_{n-1}(t)). \quad (4.10d)$$

To remove  $\alpha_{n-1}(t)$  from (4.10d), we add (4.10b) to it and get

$$2 \sum_{j=0}^{n-1} \alpha_j(t) = r_n'(t) + (4\alpha_n(t) - 2t)r_n(t) + 4n\alpha_n(t),$$

so that on account of (4.10c), equation (4.6) follows from (4.9). Equation (4.7) is true due to (4.10a)-(4.10c). In fact, from (4.10a) and (4.10b), there follows

$$\frac{r_n^2(t)}{\alpha_n(t)} = 2(n+r_n(t))\alpha_{n-1}(t) = r_n'(t) + 2(n+r_n(t))\alpha_n(t),$$

which combined with (4.10c) implies (4.7).

(ii) For  $t = \sqrt{2n} + \frac{s}{\sqrt{2n^{1/6}}}$  and  $u(s) = \lim_{n \rightarrow \infty} \sqrt{2n^{1/6}}\alpha_n(t)$ , we get

$$u'(s) = \lim_{n \rightarrow \infty} \alpha_n'(t), \quad u''(s) = \lim_{n \rightarrow \infty} \frac{\alpha_n''(t)}{\sqrt{2n^{1/6}}}.$$

If we divide both sides of (4.7) by  $\sqrt{2n^{1/6}}$  and taking  $n \rightarrow \infty$ , then we obtain

$$u''(s) = \frac{1}{2} \cdot \frac{u'(s)^2}{u(s)} + 2su(s) - 4u^2(s).$$

In addition, according to (4.6b),

$$\lim_{n \rightarrow \infty} \frac{\kappa_3(t)}{\sqrt{2n^{1/6}}} = \frac{(u'(s))^2}{4u(s)} - \frac{u'(s)}{2u(s)} - su(s) + u^2(s), \quad \lim_{n \rightarrow \infty} \frac{\kappa_4(t)}{\sqrt{2n^{1/6}}} = \frac{u'(s)}{2u(s)}.$$

For  $z = \sqrt{2n} + \frac{z^*}{\sqrt{2n^{1/6}}}$  and  $\phi(z^*) = \lim_{n \rightarrow \infty} \phi_n(z)$ , we have

$$\phi'(z^*) = \lim_{n \rightarrow \infty} \frac{\phi'_n(z)}{\sqrt{2n^{1/6}}}, \quad \phi''(z^*) = \lim_{n \rightarrow \infty} \frac{\phi''_n(z)}{2n^{1/3}}.$$

Dividing both sides of (4.6a) by  $2n^{1/3}$ , as  $n \rightarrow \infty$ , (4.8) is seen to be true.

□

**Remark 5.** Note that (4.8) is identical with (4.4). This result is mainly due to the relation between Hermite and Laguerre polynomials. Indeed, monic Hermite polynomials  $\{H_n(z)\}$  can be reduced to monic Laguerre polynomials  $\{L_n^\alpha(z)\}$  by

$$H_{2n}(z) = L_n^{(-\frac{1}{2})}(z^2), \quad H_{2n+1}(z) = L_n^{(\frac{1}{2})}(z^2).$$

Observe that  $\alpha = \pm\frac{1}{2}$  corresponds to  $\mu = 0$  (see Section 3). Write

$$\begin{aligned} z_L &:= 4n + 4^{2/3}n^{1/3}z^*, & t_L &:= 4n + 4^{2/3}n^{1/3}s, \\ z &:= \sqrt{2n} + \frac{z^*}{\sqrt{2n^{1/6}}}, & t &:= \sqrt{2n} + \frac{s}{\sqrt{2n^{1/6}}}. \end{aligned}$$

Replacing  $n$  by  $2n$  in  $z$  and  $t$ , we find

$$\begin{aligned} z^2 &= 4n + 4^{2/3}n^{1/3}z^* + \frac{(z^*)^2}{4^{2/3}n^{1/3}} \sim z_L, \\ t^2 &= 4n + 4^{2/3}n^{1/3}s + \frac{s^2}{4^{2/3}n^{1/3}} \sim t_L, \end{aligned}$$

Here the symbol  $\sim$  refers to the limiting procedure  $n \rightarrow \infty$ . The above analysis suggests that (4.4) and (4.8) are obtained by using the same scaling method.

Introducing into (4.4) the new variable  $x$  defined by  $x = -\frac{z^*-s}{r(s)}$  and the new function  $f(x) = \phi(z^*)$ , we obtain an equation in the form

$$f''(x) + p(x)f'(x) + q(x)f(x) = 0, \tag{4.11}$$

where

$$p(x) := \frac{1}{x} - \frac{1}{x-1}, \quad q(x) := \frac{a_0}{x} + \frac{a_1}{x-1} + a_2 + a_3x,$$

and

$$\begin{aligned} a_0 &= -\frac{1}{4}r'(s)^2 + \frac{1}{2}r'(s) + sr^2(s) - r^3(s), \\ a_1 &= -\frac{1}{2}r'(s), & a_2 &= -sr^2(s), & a_3 &= r^3(s). \end{aligned}$$



Note that the dependence on  $s$  of  $\{a_i\}$  is not displayed for ease of notations, in addition, the Painlevé equation

$$a_0 = -a_1^2 - a_1 - a_2 - a_3.$$

For any interval  $[c_0, x]$  excluding 0, 1, and  $\infty$ , by writing

$$f(x) = F(x) \exp \left( -\frac{1}{2} \int_{c_0}^x p(z) dz \right),$$

it follows from (4.11) that

$$F''(x) + J(x)F(x) = 0, \quad (4.12)$$

where

$$J(x) = \frac{1}{4x^2} + \frac{a_0 - \frac{1}{2}}{x} - \frac{3}{4(x-1)^2} + \frac{a_1 + \frac{1}{2}}{x-1} + a_2 + a_3x.$$

Since

$$J(x) \cong \begin{cases} \frac{1}{4x^2} + \frac{a_0 - \frac{1}{2}}{x} + a_2 - a_1 - \frac{5}{4}, & \text{as } x \rightarrow 0, \\ -\frac{3}{4(x-1)^2} + \frac{a_1 + \frac{1}{2}}{x-1} - a_1^2 - a_1 - \frac{1}{4}, & \text{as } x \rightarrow 1, \\ a_2 + a_3x, & \text{as } x \rightarrow \infty, \end{cases} \quad (4.13)$$

according to (4.12), we have the following asymptotic formulas

$$F(x) \cong \begin{cases} C_1 \frac{\sqrt{2\lambda x}}{\exp(\lambda x)} M \left( \frac{1}{2} - \frac{a_0 - \frac{1}{2}}{2\lambda}, 1, 2\lambda x \right) + C_2 \frac{\sqrt{2\lambda x}}{\exp(\lambda x)} U \left( \frac{1}{2} - \frac{a_0 - \frac{1}{2}}{2\lambda}, 1, 2\lambda x \right), & \text{as } x \rightarrow 0, \\ C_3 \frac{\exp((a_1 + \frac{1}{2})x)}{\sqrt{x-1}} + C_4 \frac{((a_1 + \frac{1}{2})x - a_1) \exp(-(a_1 + \frac{1}{2})x)}{\sqrt{x-1}}, & \text{as } x \rightarrow 1, \\ C_5 \text{Ai} \left( -\frac{a_2}{\sqrt[3]{a_3^2}} - \sqrt[3]{a_3}x \right) + C_6 \text{Bi} \left( -\frac{a_2}{\sqrt[3]{a_3^2}} - \sqrt[3]{a_3}x \right), & \text{as } x \rightarrow \infty, \end{cases}$$

where  $\lambda := \frac{1}{2}\sqrt{-4a_2 + 4a_1 + 5}$ ,  $M(\mu, \nu, z)$  and  $U(\mu, \nu, z)$  are Kummer's confluent hypergeometric functions  $M$  and  $U$  respectively,  $\text{Ai}(z)$  and  $\text{Bi}(z)$  are the Airy functions of the first and second kind respectively, and  $C_1$ - $C_6$  are arbitrary constants.

We readily see that  $x = 0$  is a regular singular point [29] for (4.11) since  $xp(x)$  and  $x^2q(x)$  are analytic at 0. From (4.11) it follows that

$$D(f) := (x^2 - x)f''(x) - f'(x) + (a_3x^3 + b_2x^2 + b_1x - a_0)f(x) = 0, \quad (4.14)$$

where

$$b_1 = -\frac{1}{4}r'(s)^2 + 2sr^2(s) - r^3(s), \quad b_2 = -sr^2(s) - r^3(s).$$

Note that

$$a_0 + a_1 - a_3 - b_1 - b_2 = 0.$$

Let

$$Y_0 = \sum_{n=0}^{\infty} c_{0,n} x^{\tau+n}, \quad c_{0,0} = 1,$$

then

$$-xD(Y_0) = \tau^2 x^\tau,$$

provided that for  $n \geq 0$

$$(\tau + n + 1)^2 c_{0,n+1} - ((\tau + n - 1)(\tau + n) - a_0) c_{0,n} - b_1 c_{0,n-1} - b_2 c_{0,n-2} - a_3 c_{0,n-3} = 0. \quad (4.15)$$

Here and in what follows  $c_{0,-3} = c_{0,-2} = c_{0,-1} := 0$ . The two solutions of (4.14) are given by

$$Y_0|_{\tau=0} = \sum_{n=0}^{\infty} c_{0,n} x^n, \quad \left. \frac{\partial Y_0}{\partial \tau} \right|_{\tau=0} = \sum_{n=1}^{\infty} d_{0,n} x^n + \ln x \cdot Y_0|_{\tau=0}$$

where

$$d_{0,n} = \left. \frac{\partial c_{0,n}}{\partial \tau} \right|_{\tau=0}.$$

See [12]. Using relation (4.15), we find  $\{c_{0,n}\}$  with  $n$  large appear in the following form

$$c_{0,n} = \sum_{\ell=3}^{\infty} \frac{\theta_{0,\ell}}{n^\ell}, \quad \theta_{0,3} := 1,$$

where the coefficients  $\{\theta_{0,\ell}\}_{\ell \geq 1}$  are determined by

$$\begin{aligned} \ell \theta_{0,\ell+3} = & \left( -\tau(1+2\ell) + \frac{1}{2}(\ell+1)\ell - a_1 \right) \theta_{0,\ell+2} \\ & + \sum_{k=3}^{\ell+1} \left\{ (-1)^{\ell-k} \left[ \binom{\ell+1}{k-3} - 2\tau \binom{\ell+1}{k-2} + \tau^2 \binom{\ell+1}{k-1} \right] \right. \\ & \left. - (b_1 + 2^{\ell-k+2} b_2 + 3^{\ell-k+2} a_3) \binom{\ell+1}{k-1} \right\} \theta_{0,k}. \end{aligned} \quad (4.16)$$

For  $\ell = 1$  the sum term is to be replaced by 0, so that

$$\theta_{0,4} = -3\tau + 1 - a_1.$$

In case  $\tau = 0$ , we obtain from (4.15)

$$(n+1)^2 c_{0,n+1} - (n(n-1) - a_0) c_{0,n} - b_1 c_{0,n-1} - b_2 c_{0,n-2} - a_3 c_{0,n-3} = 0, \quad n \geq 0,$$

and from (4.16)

$$\begin{aligned} \ell \theta_{0,\ell+3} &= \left( \frac{1}{2}(\ell+1)\ell - a_1 \right) \theta_{0,\ell+2} \\ &+ \sum_{k=3}^{\ell+1} \left\{ (-1)^{\ell-k} \binom{\ell+1}{k-3} - (b_1 + 2^{\ell-k+2} b_2 + 3^{\ell-k+2} a_3) \binom{\ell+1}{k-1} \right\} \theta_{0,k}, \quad \ell \geq 1. \end{aligned} \quad (4.17)$$

Differentiation of (4.15) and (4.16) yields, respectively,

$$\begin{aligned} \frac{\partial c_{0,n+1}}{\partial \tau} (\tau + n + 1)^2 + 2c_{0,n+1}(\tau + n + 1) - c_{0,n}(2\tau + 2n - 1) \\ - \frac{\partial c_{0,n}}{\partial \tau} ((\tau + n - 1)(\tau + n) - a_0) - \frac{\partial c_{0,n-1}}{\partial \tau} b_1 - \frac{\partial c_{0,n-2}}{\partial \tau} b_2 - \frac{\partial c_{0,n-3}}{\partial \tau} a_3 = 0, \quad n \geq 0, \end{aligned}$$

and

$$\begin{aligned} \ell \frac{\partial \theta_{0,\ell+3}}{\partial \tau} &= -(1 + 2\ell) \theta_{0,\ell+2} + \sum_{k=3}^{\ell+1} (-1)^{\ell-k} \left[ -2 \binom{\ell+1}{k-2} + 2\tau \binom{\ell+1}{k-1} \right] \theta_{0,k} \\ &+ \left( -\tau(1 + 2\ell) + \frac{1}{2}(\ell+1)\ell - a_1 \right) \frac{\partial \theta_{0,\ell+2}}{\partial \tau} \\ &+ \sum_{k=3}^{\ell+1} \left\{ (-1)^{\ell-k} \left[ \binom{\ell+1}{k-3} - 2\tau \binom{\ell+1}{k-2} + \tau^2 \binom{\ell+1}{k-1} \right] \right. \\ &\quad \left. - (b_1 + 2^{\ell-k+2} b_2 + 3^{\ell-k+2} a_3) \binom{\ell+1}{k-1} \right\} \frac{\partial \theta_{0,k}}{\partial \tau}, \quad \ell \geq 1. \end{aligned}$$

Setting  $\tau = 0$  in the above two formulas, we find

$$\begin{aligned} (n+1)^2 d_{0,n+1} + 2(n+1) c_{0,n+1} - (2n-1) c_{0,n} \\ - (n(n-1) - a_0) d_{0,n} - b_1 d_{0,n-1} - b_2 d_{0,n-2} - a_3 d_{0,n-3} = 0, \quad n \geq 0, \end{aligned}$$

and, by denoting  $\nu_{0,j} = \left. \frac{\partial \theta_{0,j}}{\partial \tau} \right|_{\tau=0}$ ,

$$\begin{aligned} \ell \nu_{0,\ell+3} &= -(1 + 2\ell) \theta_{0,\ell+2} - 2 \sum_{k=3}^{\ell+1} (-1)^{\ell-k} \binom{\ell+1}{k-2} \theta_{0,k} + \left( \frac{1}{2}(\ell+1)\ell - a_1 \right) \nu_{0,\ell+2}, \\ &+ \sum_{k=3}^{\ell+1} \left\{ (-1)^{\ell-k} \binom{\ell+1}{k-3} - (b_1 + 2^{\ell-k+2} b_2 + 3^{\ell-k+2} a_3) \binom{\ell+1}{k-1} \right\} \nu_{0,k}, \quad \ell \geq 1. \end{aligned} \quad (4.18)$$

Observe that for large  $n$

$$d_{0,n} = \sum_{\ell=4}^{\infty} \frac{\nu_{0,\ell}}{n^\ell}, \quad \nu_{0,4} = -3.$$

Now we proceed to find out the solutions of (4.11) in the form of Taylor series near  $x = 1$  which is a regular singular point since  $(x - 1)p(x)$  and  $(x - 1)^2q(x)$  are analytic at 0. Equation (4.14) can be rewritten as

$$\begin{aligned} D(f) &:= ((x - 1)^2 + (x - 1)) f''(x) - f'(x) + (a_3(x - 1)^3 + A(x - 1)^2 - a_1^2(x - 1) + a_1) f(x) \\ &= 0, \end{aligned} \tag{4.19}$$

where

$$A = -sr^2(s) + 2r^3(s).$$

Suppose

$$Y_1(x) = \sum_{n=0}^{\infty} c_{1,n}(x - 1)^{\lambda+n},$$

then

$$(x - 1)D(Y_1) = c_{1,0}\lambda(\lambda - 2)(x - 1)^{\lambda},$$

providing that

$$\begin{aligned} (\lambda^2 - 1) c_{1,1} + (a_1 + \lambda(\lambda - 1))c_{1,0} &= 0, \\ \lambda(\lambda + 2)c_{1,2} + (a_1 + \lambda(\lambda + 1))c_{1,1} - a_1^2c_{1,0} &= 0, \\ (\lambda + 1)(\lambda + 3)c_{1,3} + (a_1 + (\lambda + 1)(\lambda + 2))c_{1,2} - a_1^2c_{1,1} + Ac_{1,0} &= 0, \end{aligned} \tag{4.20}$$

and for  $n \geq 3$

$$(\lambda + n - 1)(\lambda + n + 1)c_{1,n+1} + (a_1 + (\lambda + n - 1)(\lambda + n))c_{1,n} - a_1^2c_{1,n-1} + Ac_{1,n-2} + a_3c_{1,n-3} = 0. \tag{4.21}$$

Taking  $c_{1,0} = \frac{1}{2}\lambda$ , we have

$$(x - 1)D(Y_1) = \frac{1}{2}\lambda^2(\lambda - 2)(x - 1)^{\lambda},$$

so that there are three solutions of (4.19) given by

$$Y_1|_{\lambda=2}, \quad Y_1|_{\lambda=0}, \quad \left. \frac{\partial Y_1}{\partial \lambda} \right|_{\lambda=0}.$$

Given that  $c_{1,0} = \frac{1}{2}\lambda$ , we obtain from (4.20)

$$c_{1,n}|_{\lambda=0} = 0, \quad n = 0, 1, 2, 3,$$

so that by mathematical induction it follows from (4.21) that

$$c_{1,n}|_{\lambda=0} = 0, \quad n \geq 0,$$

which implies

$$Y_1|_{\lambda=0} = 0, \quad \left. \frac{\partial Y_1}{\partial \lambda} \right|_{\lambda=0} = \sum_{n=0}^{\infty} \left. \frac{\partial c_{1,n}}{\partial \lambda} \right|_{\lambda=0} (x-1)^n.$$

Consequently, there are two solutions of (4.19) given by

$$Y_1|_{\lambda=2} = \sum_{n=0}^{\infty} (c_{1,n}|_{\lambda=2}) (x-1)^{n+2}, \quad 2 \left. \frac{\partial Y_1}{\partial \lambda} \right|_{\lambda=0} = \sum_{n=0}^{\infty} d_{1,n} (x-1)^n,$$

where

$$d_{1,n} = 2 \left. \frac{\partial c_{1,n}}{\partial \lambda} \right|_{\lambda=0}.$$

Differentiation of (4.21) gives

$$\begin{aligned} & 2(\lambda+n)c_{1,n+1} + (2(\lambda+n)-1)c_{1,n} + ((\lambda+n)^2-1) \frac{\partial c_{1,n+1}}{\partial \lambda} \\ & + (a_1 + (\lambda+n-1)(\lambda+n)) \frac{\partial c_{1,n}}{\partial \lambda} - a_1^2 \frac{\partial c_{1,n-1}}{\partial \lambda} + A \frac{\partial c_{1,n-2}}{\partial \lambda} + a_3 \frac{\partial c_{1,n-3}}{\partial \lambda} = 0. \end{aligned} \quad (4.22)$$

In particular, for  $\lambda = 0$  :

$$(n^2-1) d_{1,n+1} + (a_1 + n(n-1))d_{1,n} - a_1^2 d_{1,n-1} + A d_{1,n-2} + a_3 d_{1,n-3} = 0. \quad (4.23)$$

Choosing  $\lambda = 2$  in (4.21) leads to

$$(n+1)(n+3)c_{1,n+1} + (a_1 + (n+1)(n+2))c_{1,n} - a_1^2 c_{1,n-1} + A c_{1,n-2} + a_3 c_{1,n-3} = 0.$$

When  $n$  is large, we find from (4.21)

$$c_{1,n} = (-1)^n \sum_{\ell=1}^{\infty} \frac{\theta_{1,\ell}}{n^\ell},$$

where

$$\begin{aligned}
\theta_{1,1} &:= \frac{\lambda}{2}, \\
\theta_{1,2} &= (a_0 - \lambda(\lambda - 1)) \theta_{1,1}, \\
2\theta_{1,3} &= (a_0 - 3\lambda + 2) \theta_{1,2} - (a_1^2 + a_2 + a_3) \theta_{1,1}, \\
3\theta_{1,4} &= (a_0 + 5(1 - \lambda)) \theta_{1,3} - 2(a_1^2 + a_2 + a_3 + \lambda^2 - 3\lambda + 1) \theta_{1,2} - (a_1^2 + 4a_2 - a_3) \theta_{1,1}, \\
\ell\theta_{1,\ell+1} &= \left( a_0 + \frac{\ell(\ell+1)}{2} - (2\ell-1)\lambda - 1 \right) \theta_{1,\ell} \\
&\quad + \sum_{k=1}^{\ell-3} \left\{ (-1)^{\ell-k} \left[ \binom{\ell}{k} + (1-2\lambda) \binom{\ell-1}{k} + \lambda(\lambda-2) \binom{\ell-1}{k+1} \right] \right. \\
&\quad \quad \left. - \binom{\ell-1}{k+1} (a_1^2 + 2^{\ell-k-2}A - 3^{\ell-k-2}a_3) \right\} \theta_{1,k+2} \\
&\quad - (\ell-1) (a_1^2 + 2^{\ell-2}A - 3^{\ell-2}a_3) \theta_{1,2} - (a_1^2 + 2^{\ell-1}A - 3^{\ell-1}a_3) \theta_{1,1}, \quad \ell \geq 4.
\end{aligned}$$

Recalling  $d_{1,n} = 2 \frac{\partial c_{1,n}}{\partial \lambda} \Big|_{\lambda=0}$  and bearing in mind  $c_{1,n}|_{\lambda=0} = 0$  which suggests  $\theta_{1,\ell}|_{\lambda=0} = 0$ , we obtain for large  $n$ ,

$$d_{1,n} = (-1)^n \sum_{\ell=1}^{\infty} \frac{\nu_{1,\ell}}{n^\ell},$$

where  $\nu_{1,\ell} = 2 \frac{\partial \theta_{1,\ell}}{\partial \lambda} \Big|_{\lambda=0}$  and satisfy

$$\begin{aligned}
\nu_{1,1} &= 1, \\
\nu_{1,2} &= a_0, \\
2\nu_{1,3} &= (a_0 + 2) \nu_{1,2} - (a_1^2 + a_2 + a_3), \\
3\nu_{1,4} &= (a_0 + 5) \nu_{1,3} - 2(a_1^2 + a_2 + a_3 + 1) \nu_{1,2} - (a_1^2 + 4a_2 - a_3), \\
\ell\nu_{1,\ell+1} &= \left( a_0 + \frac{(\ell-1)(\ell+2)}{2} \right) \nu_{1,\ell} \\
&\quad + \sum_{k=1}^{\ell-3} \left\{ (-1)^{\ell-k} \left[ \binom{\ell}{k} + \binom{\ell-1}{k} \right] - \binom{\ell-1}{k+1} (a_1^2 + 2^{\ell-k-2}A - 3^{\ell-k-2}a_3) \right\} \nu_{1,k+2} \\
&\quad - (\ell-1) (a_1^2 + 2^{\ell-2}A - 3^{\ell-2}a_3) \nu_{1,2} - (a_1^2 + 2^{\ell-1}A - 3^{\ell-1}a_3), \quad \ell \geq 4.
\end{aligned}$$

In addition, for  $\lambda = 2$ ,

$$\begin{aligned}
\theta_{1,1} &= 1, \\
\theta_{1,2} &= a_0 - 2, \\
2\theta_{1,3} &= (a_0 - 4)\theta_{1,2} - (a_1^2 + a_2 + a_3) \\
3\theta_{1,4} &= (a_0 - 5)\theta_{1,3} - 2(a_1^2 + a_2 + a_3 - 1)\theta_{1,2} - (a_1^2 + 4a_2 - a_3), \\
\ell\theta_{1,\ell+1} &= \left(a_0 + \frac{\ell(\ell+1)}{2} - 4\ell + 1\right)\theta_{1,\ell} \\
&\quad + \sum_{k=1}^{\ell-3} \left\{ (-1)^{\ell-k} \left[ \binom{\ell}{k} - 3\binom{\ell-1}{k} \right] - \binom{\ell-1}{k+1} (a_1^2 + 2^{\ell-k-2}A - 3^{\ell-k-2}a_3) \right\} \theta_{1,k+2} \\
&\quad - (\ell-1)(a_1^2 + 2^{\ell-2}A - 3^{\ell-2}a_3)\theta_{1,2} - (a_1^2 + 2^{\ell-1}A - 3^{\ell-1}a_3), \quad \ell \geq 4.
\end{aligned}$$

## 5 Chazy's equations

Noticing that  $r_n(t) = \sigma'_n(t)$  is valid for both our problem and the largest eigenvalue distribution of GUE (see [10]), we can find the ODE for  $r_n(t)$  from the  $\sigma$ -form of a particular  $P_V$  satisfied by  $\sigma_n(t)$ . In general, we can develop the ODE for  $\varrho(t) := \Xi'(t)$  with  $\Xi(t)$  satisfying the  $\sigma$ -form of a general  $P_V$ . The extension of these considerations to  $P_{IV}$  is not difficult.

### 5.1 Chazy's equation for the $\sigma$ -form of $P_V$

Recall the J-M-O  $\sigma$ -form of  $P_V$  [20]

$$(t\Xi'')^2 = (\Xi - t\Xi' + 2(\Xi')^2 + (\nu_1 + \nu_2 + \nu_3)\Xi')^2 - 4\Xi'(\Xi' + \nu_1)(\Xi' + \nu_2)(\Xi' + \nu_3), \quad (5.1)$$

where  $\Xi = \Xi(t)$  and  $\nu_1, \nu_2, \nu_3$  are constants. Here we use the new notation  $\Xi$  instead of  $\sigma$ . To get the ODE for  $\varrho(t) := \Xi'(t)$ , we need to eliminate  $\Xi$  in (5.1). Differentiating both sides of (5.1), solving for  $\Xi(t)$  and plugging it back into (5.1), we obtain

$$\begin{aligned}
& (t(\varrho' + t\varrho'') + 8\varrho^3 + 6(\nu_1 + \nu_2 + \nu_3)\varrho^2 + 4(\nu_1\nu_2 + \nu_1\nu_3 + \nu_2\nu_3)\varrho + 2\nu_1\nu_2\nu_3)^2 \\
&= (4\varrho + \nu_1 + \nu_2 + \nu_3 - t)^2 \\
&\quad \cdot (t^2(\varrho')^2 + 4\varrho^4 + 4(\nu_1 + \nu_2 + \nu_3)\varrho^3 + 4(\nu_1\nu_2 + \nu_1\nu_3 + \nu_2\nu_3)\varrho^2 + 4\nu_1\nu_2\nu_3\varrho). \quad (5.2)
\end{aligned}$$

With the change of variables,

$$t = 2ie^z, \quad \vartheta(z) = -2i\varrho(t) - \frac{i}{2}(\nu_1 + \nu_2 + \nu_3),$$

where  $i^2 = -1$ , we see that  $\vartheta(z)$  satisfies

$$\left(\frac{d^2\vartheta}{dz^2} - 2\vartheta^3 - \alpha_1\vartheta - \beta_1\right)^2 = -4(\vartheta - e^z)^2 \left(\left(\frac{d\vartheta}{dz}\right)^2 - \vartheta^4 - \alpha_1\vartheta^2 - 2\beta_1\vartheta - \gamma_1\right), \quad (5.3a)$$

recognized to be the second member of the Chazy II system [11] with

$$\begin{aligned} \alpha_1 &= \frac{1}{2} (3\nu_1^2 + 3\nu_2^2 + 3\nu_3^2 - 2\nu_1\nu_2 - 2\nu_1\nu_3 - 2\nu_2\nu_3), \\ \beta_1 &= \frac{i}{2} (\nu_1 - \nu_2 - \nu_3) (\nu_1 + \nu_2 - \nu_3) (\nu_1 - \nu_2 + \nu_3), \\ \gamma_1 &= -\frac{1}{16} (\nu_1 + \nu_2 - 3\nu_3) (3\nu_1 - \nu_2 - \nu_3) (\nu_1 - 3\nu_2 + \nu_3) (\nu_1 + \nu_2 + \nu_3). \end{aligned} \quad (5.3b)$$

Now we proceed to apply our results to different ensembles.

**Example 5.1.** *For our problem, we have*

$$\Xi(t) = \sigma_n(t), \quad \varrho(t) = r_n(t).$$

*Substituting (2.13b) for  $\nu_1$ ,  $\nu_2$  and  $\nu_3$  in (5.2) and (5.3) gives rise to, respectively, the ODE for  $r_n(t)$*

$$\begin{aligned} & (t(r'_n + tr''_n) + 8r_n^3 + 6(2n + \alpha)r_n^2 + 4n(n + \alpha)r_n)^2 \\ &= (4r_n + 2n + \alpha - t)^2 (t^2(r'_n)^2 + 4r_n^4 + 4(2n + \alpha)r_n^3 + 4n(n + \alpha)r_n^2), \end{aligned} \quad (5.4)$$

*and the Chazy equation (5.3a) for*

$$\vartheta(z) = -2i r_n(t)|_{t=2ie^z} - \frac{i}{2} (2n + \alpha)$$

*with*

$$\alpha_1 = \frac{1}{2} (3\alpha^2 + 4n^2 + 4\alpha n), \quad \beta_1 = \frac{i}{2} \alpha^2 (\alpha + 2n), \quad \gamma_1 = \frac{1}{16} (2n - \alpha)(\alpha + 2n)^2 (3\alpha + 2n).$$

**Remark 6.** By means of the Riccati equations for  $R_n$  and  $r_n$ , we can derive the ODE for  $r_n$  in an alternative way which is straightforward but tedious. Solving for  $R_n$  from (2.8) yields

$$R_n = \frac{tr'_n + 2r_n^2 \pm \sqrt{\Delta_n}}{2(tr'_n - (2n + \alpha)r_n - n(n + \alpha))},$$

with  $\Delta_n := 4n(\alpha + n)r_n^2 + 4(\alpha + 2n)r_n^3 + 4r_n^4 + (tr'_n)^2$ . Choosing either sign, substituting the resulting  $R_n$  into (2.9) and clearing the square root, we obtain (5.4) again.



**Example 5.2.** *The outage probability of a single-user MIMO communication system can be calculated via the moment generating function  $\mathcal{M}(\lambda)$  of the mutual information. Under some assumptions, see [2] and [9],  $\mathcal{M}(\lambda)$  is shown to be*

$$\mathcal{M}(\lambda) = t^{-n\lambda} \frac{D_n(t, \lambda)}{D_n(t, 0)} = \exp \left( \int_{\infty}^t \frac{H_n(x) - n\lambda}{x} dx \right),$$

where

$$\begin{aligned} D_n(t, \lambda) &= \det \left( \int_0^{\infty} x^{i+j} w_{\text{dlag}}(x, t) dx \right)_{i,j=0}^{n-1}, \\ H_n(t) &:= t \frac{d}{dt} \ln D_n(t, \lambda), \end{aligned}$$

and

$$w_{\text{dlag}}(x, t) = (x + t)^{\lambda} x^{\alpha} e^{-x}.$$

Moreover, the  $\sigma$ -form of  $P_V$  with

$$\nu_1 = \lambda, \quad \nu_2 = -n, \quad \nu_3 = -n - \alpha \quad (5.5)$$

is established for

$$\Xi(t) := H_n(t) - n\lambda,$$

and the following is established

$$r_n(t) := \frac{\lambda}{h_{n-1}} \int_0^{\infty} \frac{P_n(x) P_{n-1}(x)}{x + t} w_{\text{dlag}}(x, t) dx = -\Xi'(t),$$

where

$$\int_0^{\infty} P_m(x) P_n(x) w_{\text{dlag}}(x, t) dx = h_n \delta_{mn}, \quad m \geq 0, \quad n \geq 0.$$

Plugging (5.5) into (5.2) leads to the ODE for

$$\varrho(t) = -r_n(t),$$

from which follows the Chazy's equation (5.3a) for

$$\vartheta(z) = 2i r_n(t)|_{t=2ie^z} + \frac{i}{2} (2n + \alpha - \lambda)$$

with

$$\begin{aligned}\alpha_1 &= \frac{1}{2} (4n^2 + 4n\alpha + 3\alpha^2 + 4n\lambda + 2\alpha\lambda + 3\lambda^2), \\ \beta_1 &= -\frac{i}{2}(\alpha - \lambda)(\alpha + \lambda)(2n + \alpha + \lambda), \\ \gamma_1 &= \frac{1}{16}(2n + \alpha - \lambda)(2n - \alpha + \lambda)(2n + 3\alpha + \lambda)(2n + \alpha + 3\lambda).\end{aligned}$$

**Example 5.3.** Denote by  $D_n(t)$  the determinant of the Hankel matrix generated from the moments of the time-dependent Jacobi weight

$$w(x, t) = (1 - x)^\alpha(1 + x)^\beta e^{-tx}, \quad -1 \leq x \leq 1, \quad t \in \mathbb{R},$$

namely,

$$D_n(t) = \det \left( \int_{-1}^1 x^{i+j} w(x, t) dx \right)_{i,j=0}^{n-1}.$$

It is proved in [3] that

$$\Xi(t) := t \frac{d}{dt} \ln D_n(t/2) - \frac{nt}{2} + n(n + \beta)$$

satisfies the  $\sigma$ -form of  $P_V$  with

$$\nu_1 = -\alpha, \quad \nu_2 = n, \quad \nu_3 = n + \beta. \quad (5.6)$$

Furthermore,

$$\Xi'(t) = -r_n(t/2),$$

where  $r_n(t)$  is defined by

$$r_n(t) := \frac{\alpha}{h_{n-1}} \int_{-1}^1 \frac{P_n(x)P_{n-1}(x)}{1-x} w(x, t) dx,$$

with

$$\int_{-1}^1 P_m(x)P_n(x)w(x, t)dx = h_n \delta_{mn}, \quad m \geq 0, \quad n \geq 0.$$

Substitution of (5.6) in (5.2) gives the ODE for

$$\varrho(t) = -r_n(t/2),$$

from which we obtain (5.3a) for

$$\vartheta(z) = 2i r_n(t)|_{t=iez} - \frac{i}{2} (2n - \alpha + \beta)$$

with

$$\begin{aligned}\alpha_1 &= \frac{1}{2} (4n^2 + 4n\alpha + 3\alpha^2 + 4n\beta + 2\alpha\beta + 3\beta^2), \\ \beta_1 &= -\frac{i}{2}(\alpha - \beta)(\alpha + \beta)(2n + \alpha + \beta), \\ \gamma_1 &= \frac{1}{16}(2n + \alpha - \beta)(2n - \alpha + \beta)(2n + 3\alpha + \beta)(2n + \alpha + 3\beta).\end{aligned}$$

**Example 5.4.** *The weight*

$$w(x, t) = e^{-t/x} x^\alpha (1 - x)^\beta, \quad 0 \leq x \leq 1, \quad t \geq 0$$

is studied in [5] and the *J-M-O*  $\sigma$ -form of  $P_V$  is found for

$$\Xi(t) := t \frac{d}{dt} \ln D_n(t) - n(n + \alpha + \beta)$$

with

$$\nu_1 = -(n + \alpha + \beta), \quad \nu_2 = n, \quad \nu_3 = -\beta. \quad (5.7)$$

Here again  $D_n(t)$  is the Hankel determinant

$$D_n(t) = \det \left( \int_0^1 x^{i+j} w(x, t) dx \right)_{i,j=0}^{n-1}.$$

Take into account the following result in [5]

$$\Xi'(t) = (2n + \alpha + \beta) \frac{r_n^*(t)}{t} - n$$

where

$$r_n^*(t) := \frac{t}{h_{n-1}} \int_0^1 \frac{P_n(x) P_{n-1}(x)}{x} w(x, t) dx$$

with

$$\int_0^1 P_m(x) P_n(x) w(x, t) dx = h_n \delta_{mn}, \quad m \geq 0, \quad n \geq 0,$$

then, with the aid of (5.7), we find the ODE from (5.2) for

$$\varrho(t) = (2n + \alpha + \beta) \frac{r_n^*(t)}{t} - n,$$

so that (5.3a) holds for

$$\vartheta(z) = -2i \varrho(t)|_{t=2ie^z} + \frac{i}{2} (\alpha + 2\beta)$$

with

$$\begin{aligned}\alpha_1 &= \frac{1}{2} (8n^2 + 8n\alpha + 3\alpha^2 + 8n\beta + 4\alpha\beta + 4\beta^2), \\ \beta_1 &= -\frac{i}{2} \alpha(2n + \alpha)(2n + \alpha + 2\beta), \\ \gamma_1 &= -\frac{1}{16} (\alpha - 2\beta)(\alpha + 2\beta)(4n + \alpha + 2\beta)(4n + 3\alpha + 2\beta).\end{aligned}$$

## 5.2 Chazy's equation for the $\sigma$ -form of $P_{IV}$

We go ahead to deal with the  $\sigma$ -form of  $P_{IV}$  in the spirit of the previous section. The J-M-O  $\sigma$ -form of  $P_{IV}$  [20] is given by

$$(\Xi'')^2 = 4(t\Xi' - \Xi)^2 - 4\Xi'(\Xi' + \nu_1)(\Xi' + \nu_2),$$

where  $\Xi = \Xi(t)$  and  $\nu_1, \nu_2$  are constants. From this equation follows the ODE for  $\varrho(t) := \Xi'(t)$ ,

$$(\varrho'' + 6\varrho^2 + 4(\nu_1 + \nu_2)\varrho + 2\nu_1\nu_2)^2 = 4t^2((\varrho')^2 + 4\varrho^3 + 4(\nu_1 + \nu_2)\varrho^2 + 4\nu_1\nu_2\varrho). \quad (5.8)$$

Setting

$$\vartheta(z) = -\frac{1}{2} \varrho(t)|_{t=\frac{z}{\sqrt{2}}} - \frac{1}{6}(\nu_1 + \nu_2),$$

we find

$$(\vartheta''(z) - 6\vartheta(z)^2 - \alpha_1)^2 = z^2(\vartheta'(z)^2 - 4\vartheta(z)^3 - 2\alpha_1\vartheta(z) - \beta_1), \quad (5.9a)$$

which is the first member of the Chazy II system [11] with

$$\alpha_1 = \frac{1}{6}(-\nu_1^2 + \nu_1\nu_2 - \nu_2^2), \quad \beta_1 = -\frac{1}{54}(\nu_1 - 2\nu_2)(2\nu_1 - \nu_2)(\nu_1 + \nu_2). \quad (5.9b)$$

Now we intend to show the application of the above result.

**Example 5.5.** *Regarding the deformed Hermite weight with one jump,*

$$w(x; t) = e^{-x^2} \left( 1 - \frac{\beta}{2} + \beta\theta(x - t) \right) = \begin{cases} (1 + \frac{\beta}{2}) e^{-x^2}, & \text{if } x > t \\ (1 - \frac{\beta}{2}) e^{-x^2}, & \text{if } x \leq t \end{cases},$$

the following formulas are established in [10] (The  $\tilde{x}$  there has been replaced here with  $t$ ),

$$\begin{aligned}r_n^2(t) &= 2(n + r_n(t))\alpha_n\alpha_{n-1}, \\ r'_n(t) &= 2(n + r_n(t))(\alpha_{n-1} - \alpha_n), \\ \frac{d}{dt} \ln D_n(t) &= 2tr_n(t) - 2(n + r_n(t))(\alpha_n + \alpha_{n-1}), \\ \frac{d^2}{dt^2} \ln D_n(t) &= 2r_n(t).\end{aligned} \quad (5.10)$$

Recall here that

$$\begin{aligned} r_n(t) &= \beta \frac{P_n(t, t) P_{n-1}(t, t)}{h_{n-1}} e^{-t^2}, \\ D_n(t) &= \det \left( \int_{-\infty}^{\infty} x^{i+j} w(x; t) dx \right)_{i,j=0}^{n-1}, \end{aligned}$$

with

$$\begin{aligned} h_n \delta_{mn} &= \int_{-\infty}^{\infty} P_m(x) P_n(x) w(x; t) dx, \\ z P_n(z) &= P_{n+1}(z) + \alpha_n P_n(z) + \beta_n P_{n-1}(z). \end{aligned}$$

From (5.10), we can derive the  $\sigma$ -form of  $P_{IV}$  for

$$\Xi(t) := \frac{d}{dt} \ln D_n(t)$$

with

$$\nu_1 = 0, \quad \nu_2 = 2n. \quad (5.11)$$

Hence the the ODE for

$$\varrho(t) = r_n(t)$$

follows immediately from (5.8) and, as a consequence, the Chazy's equation (5.9a) holds for

$$v(z) = -\frac{1}{2} r_n(t)|_{t=\frac{z}{\sqrt{2}}} - \frac{n}{3}$$

with

$$\alpha_1 = -\frac{2}{3}n^2, \quad \beta_1 = -\frac{8}{27}n^3. \quad (5.12)$$

**Remark 7.** Denote by  $\mathbb{P}_{\max}(n, t)$  the largest eigenvalue distribution of GUE on  $(0, t)$  and by  $\mathbb{P}_{\min}(n, t)$  the smallest one on  $(t, \infty)$ . Then we readily see in this last example that

$$\Xi(t) = \frac{d}{dt} \ln \mathbb{P}_{\max}(n, t) \quad \text{or} \quad -\frac{d}{dt} \ln \mathbb{P}_{\min}(n, t)$$

according as

$$\beta = -2 \quad \text{or} \quad 2.$$

Indeed, for instance, for  $\beta = -2$ , we have

$$\begin{aligned}\bar{D}_n(t) &:= \det \left( \int_{-\infty}^t x^{i+j} e^{-x^2} dx \right)_{i,j=0}^{n-1} = \frac{D_n(t)}{2^n}, \\ \frac{\bar{D}_n(t)}{\bar{D}_n(\infty)} &= \mathbb{P}_{\max}(n, t),\end{aligned}$$

so that

$$\frac{d}{dt} \ln \mathbb{P}_{\max}(n, t) = \frac{d}{dt} \ln \bar{D}_n(t) = \frac{d}{dt} \ln D_n(t) = \Xi(t).$$

In addition, we notice that

$$r_n(t) = -\frac{P_n(t, t)P_{n-1}(t, t)}{\hbar_{n-1}}e^{-t^2},$$

with

$$\int_{-\infty}^t P_m(x)P_n(x)e^{-x^2}dx = \frac{1}{2}h_n\delta_{mn} =: \hbar_n\delta_{mn}.$$

**Example 5.6.** The  $\sigma$ -form of  $P_{IV}$  with parameters given by (5.11) is also satisfied by

$$\Xi(t) := \frac{d}{dt} \ln \mathbb{P}(n, t),$$

where  $\mathbb{P}(n, t)$  is the probability that  $(-t, t)$  is free of eigenvalues in GUE. See [4] and [27]. Since

$$r_n(t) := 2\frac{P_n(t, t)P_{n-1}(t, t)}{\hbar_{n-1}}e^{-t^2} = \frac{\Xi'(t)}{2} = \frac{\varrho(t)}{2},$$

equation (5.9a) is valid for

$$v(z) = -\varrho(t)|_{t=\frac{z}{\sqrt{2}}} - \frac{n}{3}$$

with parameters given by (5.12). This result is in agreement with the one in [4] found by using the Riccati equations for  $r_n$  and  $R_n$ .

**Acknowledgements.** The financial support of the Macau Science and Technology Development Fund under grant number FDCT 077/2012/A3, FDCT 130/2014/A3 is gratefully acknowledged. We also like to thank the University of Macau for generous support: MYRG 2014–00011 FST, MYRG 2014–00004 FST.

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